ME243B: Course Project

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Course ME243B **Project** Final Project

LQG Design

Starting with the system G_{bd},

$$\frac{d}{dt}\begin{bmatrix} u \\ v \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ \mathcal{B}_1 & \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \\ \dot{v} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \mathcal{B}_2 \end{bmatrix}}_{B} \dot{u}$$

$$\dot{y} = \underbrace{\begin{bmatrix} \mathcal{D} & \mathcal{C} & 0 \end{bmatrix}}_{C} \begin{bmatrix} u \\ v \\ \dot{v} \end{bmatrix} + \underbrace{0}_{D} \dot{u}.$$

We can include disturbance and noise to this model.

$$\begin{aligned} \dot{\psi} &= A\psi + B(u+d) \\ y &= C\psi + n \end{aligned}$$

We want to curate the disturbance with a shaping filter:

$$d = W_d \, \underline{d} = \frac{1}{\tau_d s + 1} \, \underline{d},$$

Then to get this into state space:

$$\begin{aligned} d\tau_d s + d &= \underline{d} \\ \dot{d} &= -\frac{1}{\tau_d} \, d + \frac{1}{\tau_d} \, \underline{d} \end{aligned}$$

We can define the state as $x_d = d$ to get:

$$\begin{aligned} \dot{x}_d &= \overbrace{-\frac{1}{\tau_d}}^{A_d} \, x_d + \overbrace{\frac{1}{\tau_d}}^{B_d} \, \underline{d} \\ d &= \underbrace{1}_{C_d} \, x_d \end{aligned}$$

Which lets us define a new state $\phi^T = \begin{bmatrix} \psi^T & x_d \end{bmatrix}$

$$\dot{\phi} = \overbrace{\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix}}^{\hat{A}} \phi + \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{\hat{B}_1} u + \overbrace{\begin{bmatrix} 0 \\ B_d \end{bmatrix}}^{\hat{B}_2} \underline{d} \tag{1}$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{\hat{C}} \phi + n \tag{2}$$

We can use the `lqr` function in Matlab to get the optimal gain that minimizes our cost function

$$K = \texttt{lqr}(\hat{A},\ \hat{B}_1,\ \hat{C}^T\hat{C},\ 10^{-3})$$

Then use `kalman` to get the optimal estimator

$$L = \texttt{kalman}\left(\texttt{ss}\left(\hat{A}, \begin{bmatrix} \hat{B}_1 & \hat{B}_2 \end{bmatrix}, \hat{C}, \begin{bmatrix} 0 & 0 \end{bmatrix}\right),\ 1, 10^{-8}\right)$$

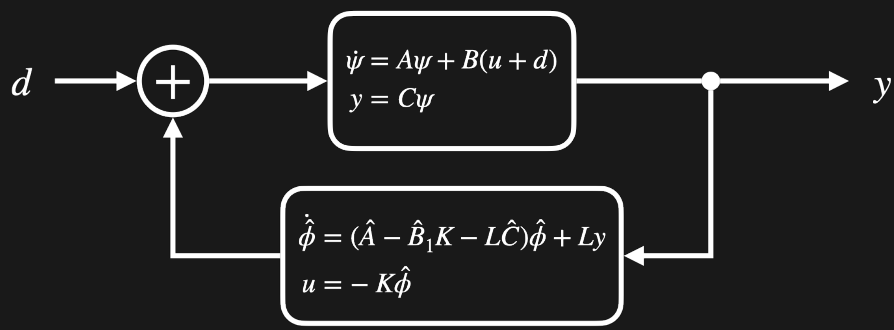


Figure 1: Combining the estimator and LQR controller, we get a LQG controller and put it in feedback with the original system to get the transfer function from d to y.

Reading off the figure, we can write down the state space model for our block diagram

$$\begin{bmatrix} \dot{\psi} \\ \dot{\hat{\phi}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & \hat{A} - \hat{B}_1 K - L\hat{C} \end{bmatrix} \begin{bmatrix} \psi \\ \hat{\phi} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} d,$$
$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \hat{\phi} \end{bmatrix}.$$

Therefore,

$$G_{d\to y}(s) = \begin{bmatrix} C & 0 \end{bmatrix} \left(sI - \begin{bmatrix} A & -BK \\ LC & \hat{A} - \hat{B}_1 K - L\hat{C} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

Plotting the bode diagram from this transfer function gives us:

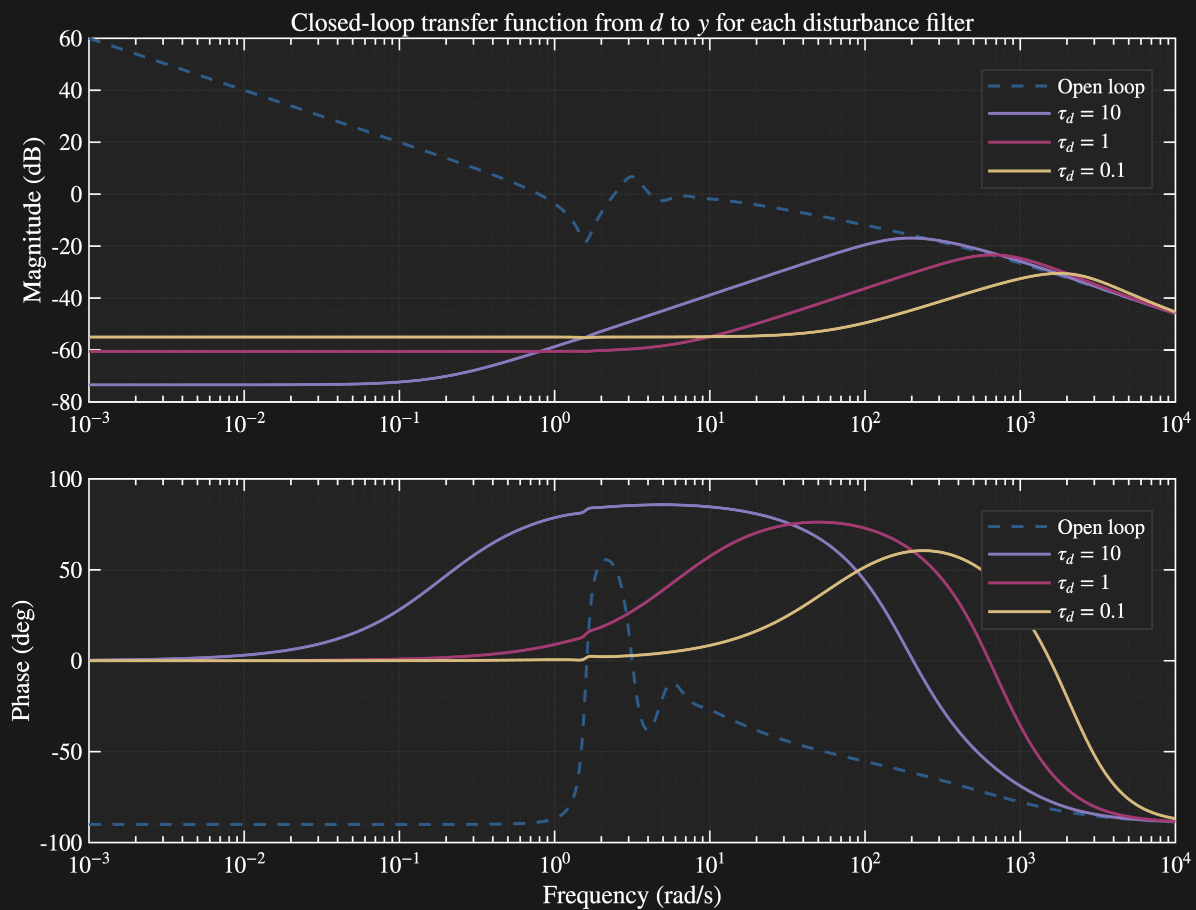


Figure 2: As τ_d is decreased, the Kalman filter must account for more high frequency noise and we see a corresponding shift in the bode plot. This is much better than the open loop system gives huge amplification to low frequency disturbances.

Controller Reduction

Algorithm

The construction is very straightforward. For each order, r :

1. Form reduced system G_r with `balred`$(G_{\text{bd}},\ r)$
2. Form filter system $\hat{G}_r$ as in (1, 2) using G_r state space matrices rather than G_{bd}
3. Design LQG controller with $\hat{G}_r$ system
4. Plot transfer function $d \to y$ for each τ_d

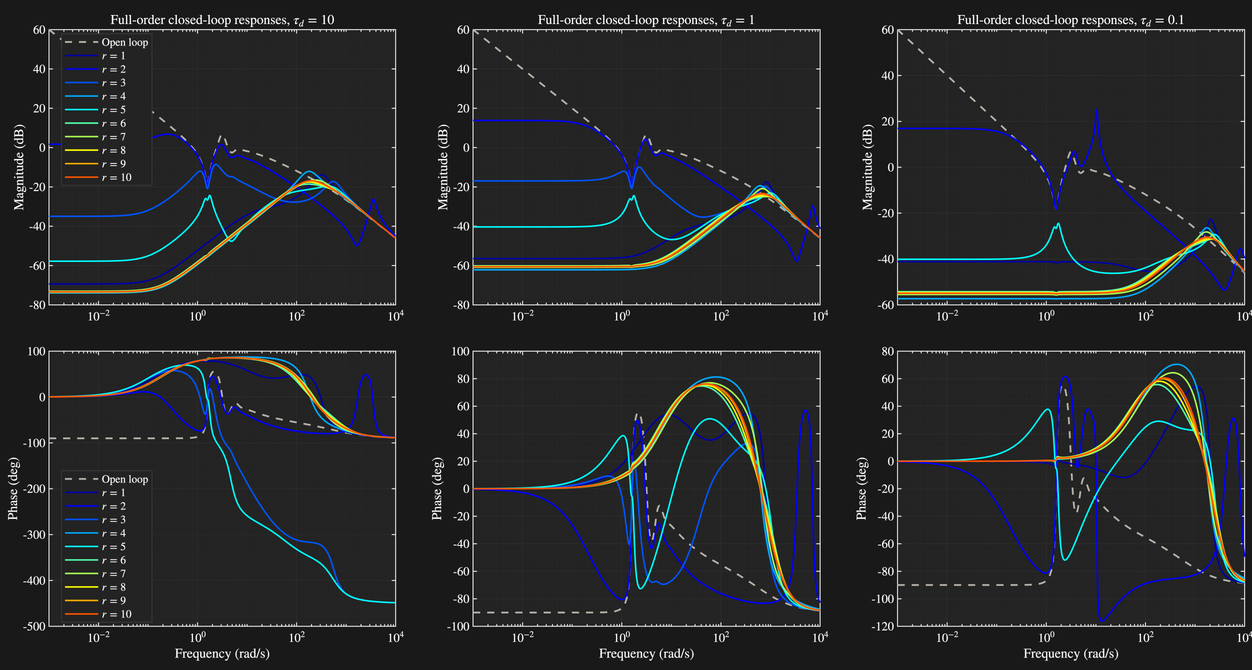


Figure 3: Closed-loop frequency responses for $\tau_d = 10$, 1, and 0.1, respectively. (Larger figures appended for easier viewing.)

We found one unstable case: r = 3, $\tau_d = 0.1$. This doesn't mean there is no controller of order 3 that is stabilizing; it just means that the construction in `balredd` does not give a stabilizing controller. The reason almost all of these controllers are stabilizing is because of the high damping. On the next page, we plot a low damping $\beta = 0.001$ instead of $\beta = 0.1$.

Table 1: Closed-loop stability for each reduced model order and disturbance model for low damping.

$\tau_d \backslash r$	1	2	3	4	5	6	7	8	9	10
10	**S**	**U**	**S**	**S**	**S**	**S**	**S**	**S**	**S**	**S**
1	**S**	**U**	**S**	**U**	**S**	**U**	**S**	**U**	**S**	**U**
0.1	**S**	**U**	**S**	**U**	**S**	**U**	**S**	**U**	**S**	**U**

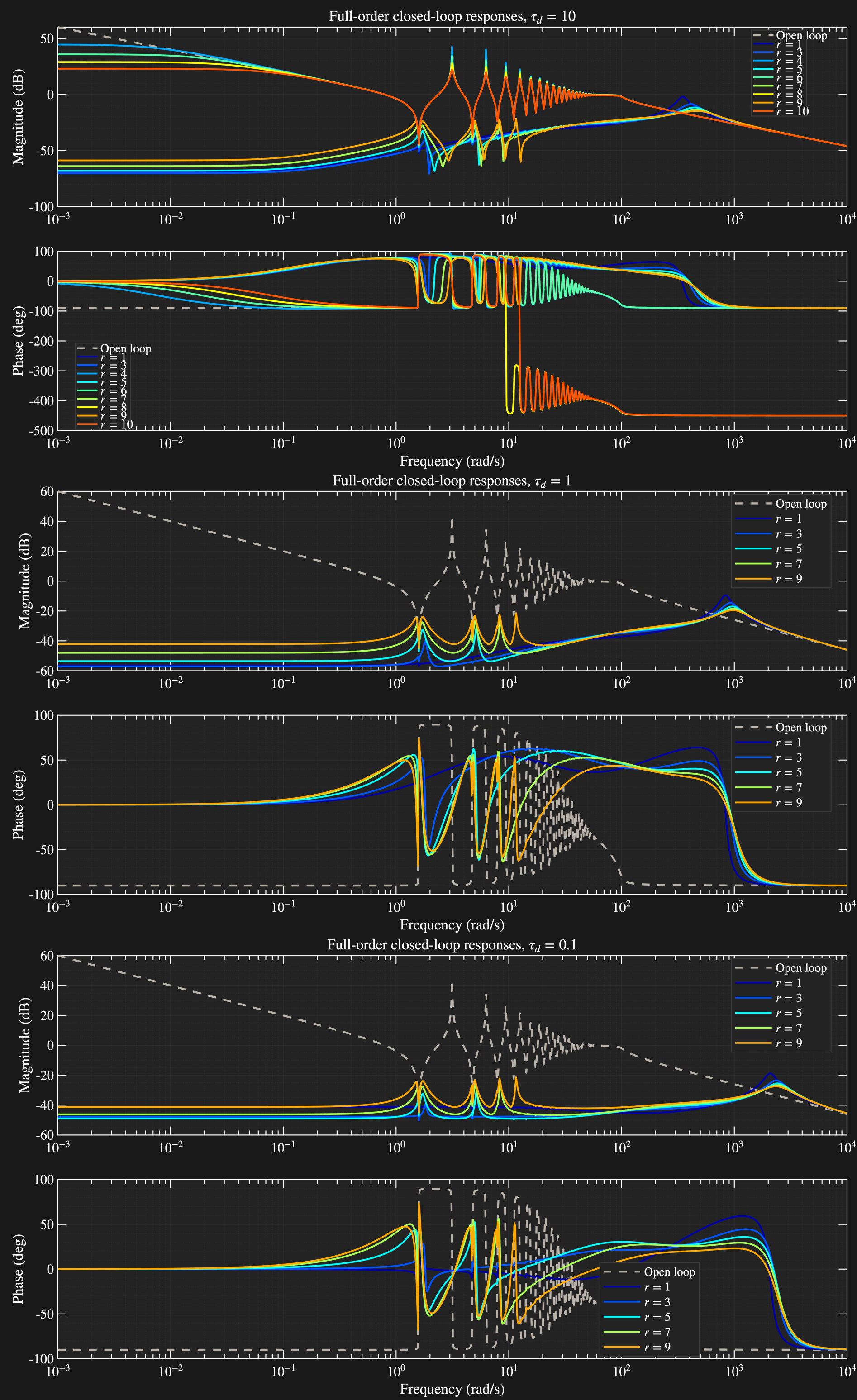
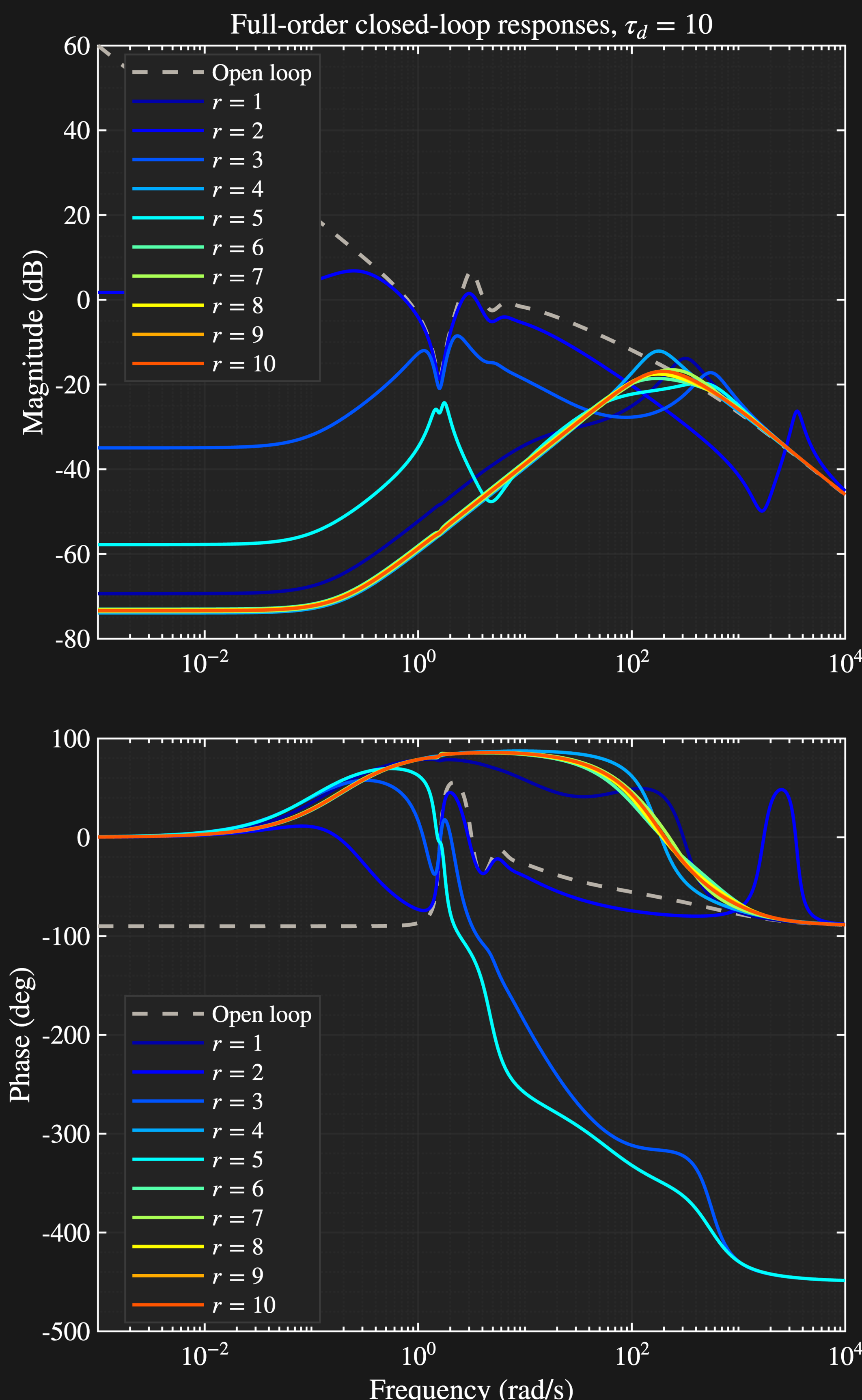
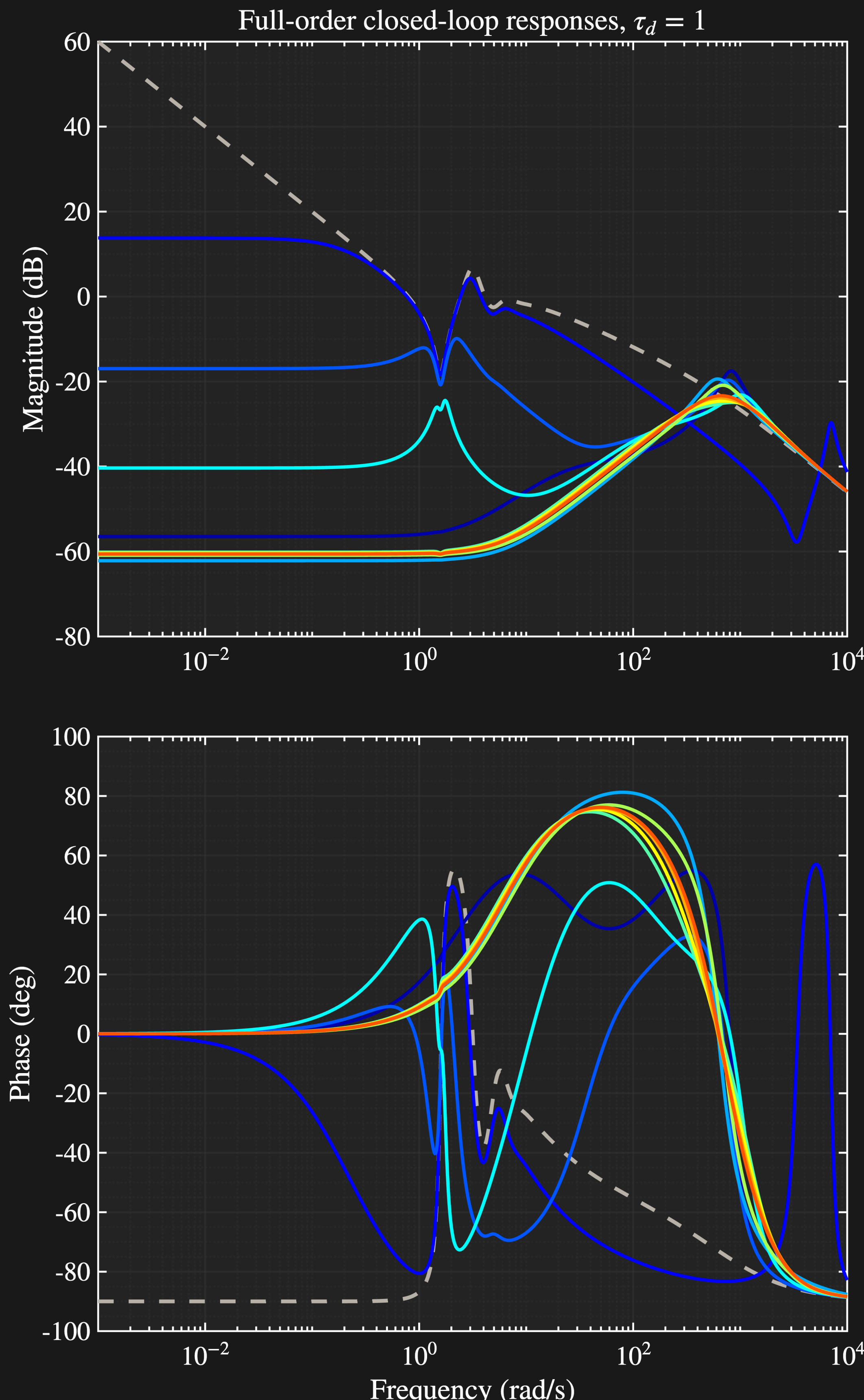
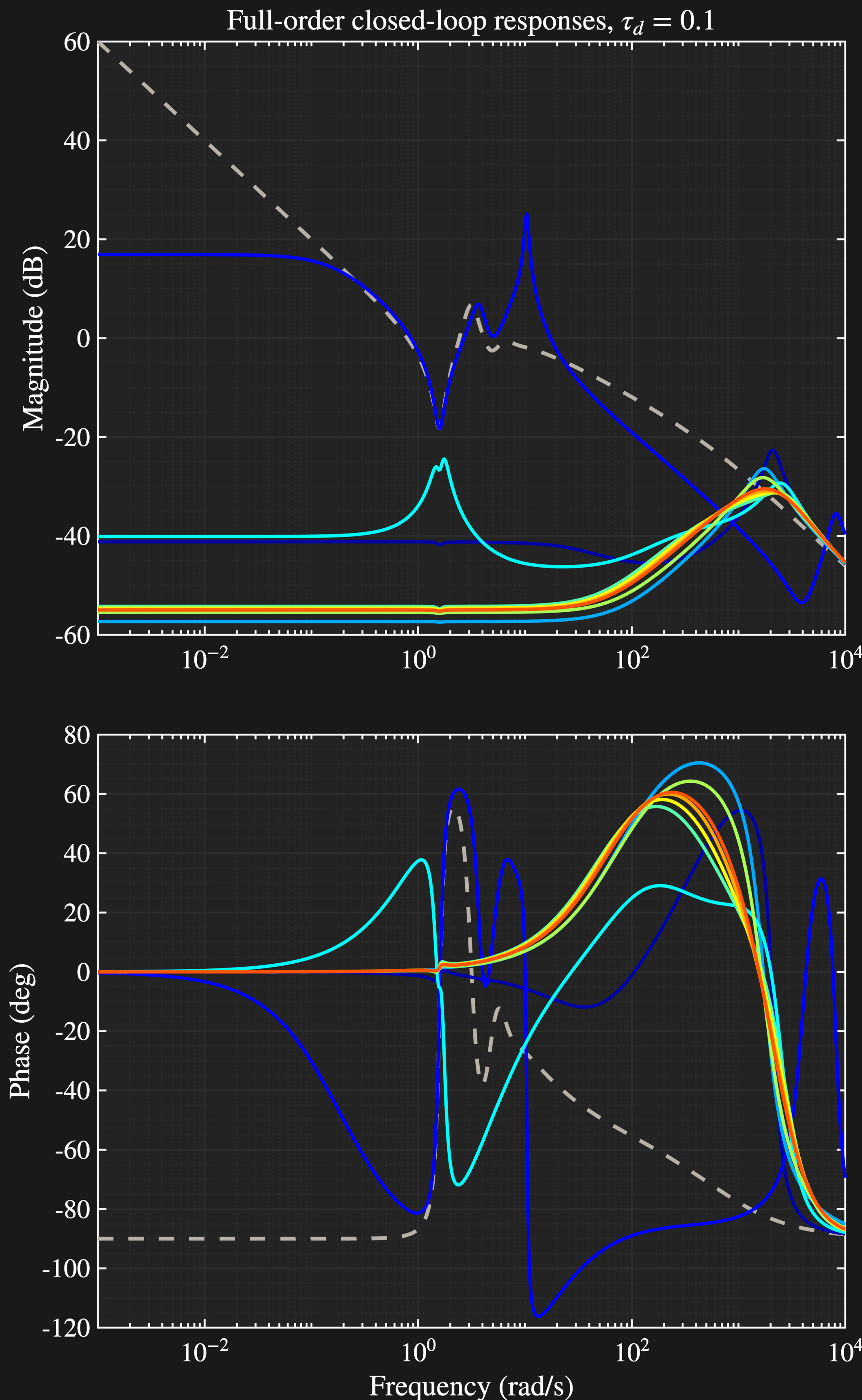


Figure 4: Closed-loop frequency responses for $\tau_d = 10$, 1, and 0.1, with $\beta = 0.001$. With the filters accounting for higher frequencies, we see a 'desirable' lower amplification of disturbance. In conjunction we also have fewer reduced models that are capable of stabilizing the plant.



Full-order closed-loop responses, $\tau_d = 10$
Magnitude (dB)
60
40
20
0
-20
-40
-60
-80
Open loop
$r = 1$
$r = 2$
$r = 3$
$r = 4$
$r = 5$
$r = 6$
$r = 7$
$r = 8$
$r = 9$
$r = 10$
10^{-2}
10^{0}
10^{2}
10^{4}
Phase (deg)
100
0
-100
-200
-300
-400
-500
Open loop
$r = 1$
$r = 2$
$r = 3$
$r = 4$
$r = 5$
$r = 6$
$r = 7$
$r = 8$
$r = 9$
$r = 10$
10^{-2}
10^{0}
10^{2}
10^{4}
Frequency (rad/s)


Full-order closed-loop responses, $\tau_d = 1$
Magnitude (dB)
60
40
20
0
-20
-40
-60
-80
10^{-2}
10^{0}
10^{2}
10^{4}
Phase (deg)
100
80
60
40
20
0
-20
-40
-60
-80
-100
10^{-2}
10^{0}
10^{2}
10^{4}
Frequency (rad/s)



Full-order closed-loop responses, $\tau_d = 0.1$
Magnitude (dB)
60
40
20
0
-20
-40
-60
10^{-2}
10^{0}
10^{2}
10^{4}
Phase (deg)
80
60
40
20
0
-20
-40
-60
-80
-100
-120
10^{-2}
10^{0}
10^{2}
10^{4}
Frequency (rad/s)